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Parity-dependent squeezing of light

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Abstract. A parity-dependent squeezing operator is introduced which imposes different $SU(1,1)$ rotations on the even and odd subspaces of the harmonic oscillator Hilbert space. This operator is used to define parity-dependent squeezed states which exhibit highly non-classical properties such as strong antibunching, quadrature squeezing, strong oscillations in the photon-number distribution, etc. In contrast to the usual squeezed states whose Q and Wigner functions are simply Gaussians, the parity-dependent squeezed states have much more complicated Q and Wigner functions that exhibit an interesting interference in phase space. The generation of these states by parity-dependent quadratic Hamiltonians is also discussed.

1. Introduction

Squeezed states have been studied extensively in the last few years [1–3]. They exhibit non-classical behaviour such as oscillations in the photon-number distribution [5], sub-Poissonian photon statistics (antibunching), reduction of quantum fluctuations in either of the field quadratures (quadrature squeezing), etc. Squeezing of the two-mode light field was studied in [2], and squeezing criteria for multi-mode systems were introduced in [4]. It was pointed out [6] that the usual type of two-mode squeezing defined in [2] is based on reducible representations of the $SU(1,1)$ group. A more general kind of two-mode squeezing was considered [6] where different $SU(1,1)$ rotations are imposed on each irreducible sector. It was shown [6] that the two-mode squeezed states produced by these generalized squeezing transformations have interesting properties.

In the present paper we study a similar generalization of squeezing for the single-mode light field. The ordinary single-mode squeezed states are produced by unitary transformations belonging to a reducible representation of $SU(1,1)$. Such reducible representations contain two irreducible components: the first is the representation that acts on the 'even Fock subspace' (i.e. the subspace spanned by the even number eigenstates) and the second is the representation that acts on the 'odd Fock subspace' (i.e. the subspace spanned by the odd number eigenstates). In terms of the Bargmann index k that labels unitary irreducible representations of $SU(1,1)$ [7], the even Fock subspace corresponds to the representation with $k = \frac{1}{4}$, while the odd Fock subspace corresponds to the case $k = \frac{3}{4}$. We impose different $SU(1,1)$ rotations on the two irreducible sectors and thus introduce states whose even and odd components are squeezed with different squeezing parameters. We refer to these states as the parity-dependent squeezed states.

The parity-dependent squeezed states exhibit highly non-classical behaviour. They are characterized by more squeezing parameters than the ordinary squeezed states, and we

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find regions of these parameters where there are strong oscillations in the photon-number distribution, strong antibunching, quadrature squeezing, etc. We demonstrate that these states can be more strongly antibunched than the ordinary squeezed states. A further interesting feature of the parity-dependent squeezed states is that their Q and Wigner functions are not necessarily Gaussians. The examples that we consider show a very strong interference in phase space.

2. Parity-dependent squeezed states

2.1. Definitions and basic properties

We consider the harmonic oscillator Hilbert space $\mathcal{H}$ and express it as the direct sum

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \tag{2.1}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are the subspaces spanned by the even and odd number eigenstates, respectively,

$$\begin{aligned} \mathcal{H}_0 &= \{|2n\rangle; n = 0, 1, 2, \ldots\} \\ \mathcal{H}_1 &= \{|2n+1\rangle; n = 0, 1, 2, \ldots\}\,. \end{aligned} \tag{2.2}$$

The projection operators onto these Hilbert spaces are given by

$$\Pi_0 = \sum_{n=0}^{\infty} |2n\rangle\langle 2n| \qquad \Pi_1 = \sum_{n=0}^{\infty} |2n+1\rangle\langle 2n+1|\,. \tag{2.3}$$

They have the usual properties of projection operators:

$$\Pi_0 + \Pi_1 = 1 \qquad \Pi_i \Pi_j = \Pi_i \delta_{ij} \qquad i, j = 0, 1\,. \tag{2.4}$$

The parity operator is given by

$$P = \Pi_0 - \Pi_1 = \exp(\mathrm{i}\pi a^\dagger a) \tag{2.5}$$

where a and $a^\dagger$ are the boson annihilation and creation operators. The properties of the parity operator are

$$P^2 = 1 \qquad Pa = -aP \qquad Pa^\dagger = -a^\dagger P\,. \tag{2.6}$$

If $|\Psi\rangle$ is an arbitrary state in the Hilbert space $\mathcal{H}$ with the position-representation wavefunction $\Psi(x) = \langle x|\Psi\rangle$, then the action of the parity operator is the inversion:

$$P\Psi(x) = \langle x|P|\Psi\rangle = \Psi(-x)\,. \tag{2.7}$$

The same property also holds for the momentum representation.

The ordinary single-mode squeezing operator is defined as [1]

$$\begin{aligned} S(\xi,\lambda) &= \exp\left(\xi K_+ - \xi^* K_-\right) \exp(2\mathrm{i}\lambda K_0) \\ \xi &= -r\exp(-\mathrm{i}\theta)\,. \end{aligned} \tag{2.8}$$

The parameters r, θ and λ are real. The operators

$$K_0 = \tfrac{1}{4}(aa^\dagger + a^\dagger a) \qquad K_+ = \tfrac{1}{2}a^{\dagger 2} \qquad K_- = \tfrac{1}{2}a^2 \tag{2.9}$$

form the single-mode bosonic realization of the $SU(1,1)$ Lie algebra:

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0\,. \tag{2.10}$$

The Casimir operator is

$$\begin{aligned} K^2 &= K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) \\ &= k(k-1) = -\tfrac{3}{16} \end{aligned} \tag{2.11}$$

where the Bargmann index k labels the irreducible representations of $SU(1,1)$ [7]. In the present case k can acquire two values: $\frac{1}{4}$ and $\frac{3}{4}$; so we have two irreducible representations. The even subspace $\mathcal{H}_0$ corresponds to the representation with $k=\frac{1}{4}$, and the odd subspace $\mathcal{H}_1$ corresponds to the case $k=\frac{3}{4}$. The unitary squeezing operators $S(\xi,\lambda)$ of equation (2.8) form a reducible representation since they act on both irreducible sectors. More specifically, they form the $k=\frac{1}{4}$ irreducible representation when they act on $\mathcal{H}_0$ only and the $k=\frac{3}{4}$ irreducible representation when they act on $\mathcal{H}_1$ only. Related to this is the fact that

$$[S(\xi,\lambda),\Pi_0]=[S(\xi,\lambda),\Pi_1]=0\,. \tag{2.12}$$

The parity-dependent squeezing operator is defined as

$$\begin{aligned} &U(\xi_0,\lambda_0;\xi_1,\lambda_1)=S(\xi_0,\lambda_0)\Pi_0+S(\xi_1,\lambda_1)\Pi_1 \\ &\xi_j=-r_j\exp(-\mathrm{i}\theta_j)\,j=0,1\,. \end{aligned} \tag{2.13}$$

This is a generalization of the ordinary squeezing operator (2.8). Only in the special case

$$r_0=r_1 \qquad \theta_0=\theta_1 \qquad \lambda_0=\lambda_1 \tag{2.14}$$

does the operator (2.13) reduce to the operator (2.8). The parity-dependent squeezing operator squeezes each irreducible sector independently. Acting with this operator on the Glauber coherent state [8]

$$|\beta\rangle=\mathrm{e}^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle \tag{2.15}$$

we obtain the parity-dependent squeezed state:

$$|\beta;\xi_0,\lambda_0;\xi_1,\lambda_1\rangle=U(\xi_0,\lambda_0;\xi_1,\lambda_1)|\beta\rangle=S(\xi_0,\lambda_0)\Pi_0|\beta\rangle+S(\xi_1,\lambda_1)\Pi_1|\beta\rangle\,. \tag{2.16}$$

In the special case (2.14), this state reduces to the ordinary squeezed state [1]. Note that the states $\Pi_0|\beta\rangle$ and $\Pi_1|\beta\rangle$ (with a suitable normalization) are the even and odd coherent states [9], which are special cases of macroscopic quantum superpositions also known as the Schrödinger-cat states [10]. We see that the parity-dependent squeezed states $|\beta;\xi_0,\lambda_0;\xi_1,\lambda_1\rangle$ can be viewed as superpositions of two differently squeezed Schrödinger-cat states.

The overlap of two parity-dependent squeezed states with the same squeezing parameters and different coherent amplitudes is

$$\langle\alpha;\xi_0,\lambda_0;\xi_1,\lambda_1|\beta;\xi_0,\lambda_0;\xi_1,\lambda_1\rangle=\langle\alpha|\beta\rangle=\exp\left(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2+\alpha^*\beta\right) \tag{2.17}$$

where we have used the unitarity of the parity-dependent squeezing operator. We also multiply the identity resolution [8]

$$\frac{1}{\pi}\int\mathrm{d}^2\beta\,|\beta\rangle\langle\beta|=1 \tag{2.18}$$

by the operator $U(\xi_0,\lambda_0;\xi_1,\lambda_1)$ on the left and by its Hermitian conjugate $U^\dagger(\xi_0,\lambda_0;\xi_1,\lambda_1)$ on the right in order to prove another form of the identity resolution:

$$\frac{1}{\pi}\int\mathrm{d}^2\beta\,|\beta;\xi_0,\lambda_0;\xi_1,\lambda_1\rangle\langle\beta;\xi_0,\lambda_0;\xi_1,\lambda_1|=1\,. \tag{2.19}$$

Equations (2.17) and (2.19) show that the set of the states $\{|\beta;\xi_0,\lambda_0;\xi_1,\lambda_1\rangle\}$ with fixed squeezing parameters and all the complex numbers β forms an overcomplete basis in the Hilbert space $\mathcal{H}$.

2.2. Parity-dependent Bogoliubov transformations

We introduce the parity-dependent Bogoliubov transformations:

$$\begin{aligned} b &\equiv U(\xi_0, \lambda_0; \xi_1, \lambda_1) a U^\dagger(\xi_0, \lambda_0; \xi_1, \lambda_1) \\ b^\dagger &\equiv U(\xi_0, \lambda_0; \xi_1, \lambda_1) a^\dagger U^\dagger(\xi_0, \lambda_0; \xi_1, \lambda_1)\,. \end{aligned} \tag{2.20}$$

The commutation relation is preserved for the parity-dependent Bogoliubov quasiparticles:

$$[b, b^\dagger] = [a, a^\dagger] = 1\,. \tag{2.21}$$

In the special case (2.14), the operators b and $b^\dagger$ are linear combinations of a and $a^\dagger$:

$$b = \mu_j a + \nu_j a^\dagger \qquad b^\dagger = \nu_j^* a + \mu_j^* a^\dagger \tag{2.22}$$

where j is either 0 or 1, and we use the notation

$$\begin{aligned} &\mu_j \equiv \cosh r_j \exp(-\mathrm{i}\lambda_j) \\ &\nu_j \equiv \sinh r_j \exp[-\mathrm{i}(\theta_j + \lambda_j)] \\ &|\mu_j|^2 - |\nu_j|^2 = 1\,. \end{aligned} \tag{2.23}$$

However, in general the transformation (2.20) is much more complicated and b and $b^\dagger$ are not linear combinations of a and $a^\dagger$. Using equation (2.20), we easily prove

$$U(\xi_0, \lambda_0; \xi_1, \lambda_1) f(a, a^\dagger) U^\dagger(\xi_0, \lambda_0; \xi_1, \lambda_1) = f(b, b^\dagger)\,. \tag{2.24}$$

It is easily seen that the parity-dependent squeezed states of (2.16) are the ordinary coherent states with respect to the Bogoliubov quasiparticles. For example, they are eigenstates of b:

$$b|\beta; \xi_0, \lambda_0; \xi_1, \lambda_1\rangle = \beta|\beta; \xi_0, \lambda_0; \xi_1, \lambda_1\rangle\,. \tag{2.25}$$

We can also introduce the 'b-position' operator

$$x_b = U(\xi_0, \lambda_0; \xi_1, \lambda_1) x U^\dagger(\xi_0, \lambda_0; \xi_1, \lambda_1) = \frac{b + b^\dagger}{\sqrt{2}} \tag{2.26}$$

whose eigenstates are the 'b-position' states:

$$|x\rangle_b = U(\xi_0, \lambda_0; \xi_1, \lambda_1)|x\rangle \qquad x_b|x\rangle_b = x|x\rangle_b\,. \tag{2.27}$$

The overlap of the parity-dependent squeezed states with the 'b-position' states is a simple Gaussian:

$${}_b\langle x|\beta; \xi_0, \lambda_0; \xi_1, \lambda_1\rangle = \langle x|\beta\rangle = \pi^{-1/4} \exp\left[-\frac{\beta}{2}(\beta^* - \beta) - \left(\beta - \frac{x}{\sqrt{2}}\right)^2\right]. \tag{2.28}$$

Consequently, the variances of the 'b-position' and 'b-momentum' over the parity-dependent squeezed states are

$$(\Delta x_b)^2 = (\Delta p_b)^2 = \tfrac{1}{2}\,. \tag{2.29}$$

The uncertainties with respect to the ordinary position and momentum are discussed in section 3.2.

2.3. The parity-dependent Hamiltonian

Here we give a Hamiltonian that can produce the parity-dependent squeezed states. Using the identity

$$\exp(A\Pi_0 + B\Pi_1) = \Pi_0 \exp(A) + \Pi_1 \exp(B) \tag{2.30}$$

(where A and B are operators that commute with Π_0 and Π_1), we easily see that quantum systems governed by the Hamiltonian

$$H = \omega a^\dagger a + \Pi_0(g_0 a^{\dagger 2} + g_0^* a^2) + \Pi_1(g_1 a^{\dagger 2} + g_1^* a^2) \tag{2.31}$$

will evolve ordinary coherent states $|\beta\rangle$ into the parity-dependent squeezed states of equation (2.16). From the relations $\Pi_0 = (1+P)/2$ and $\Pi_1 = (1-P)/2$, we see that the Hamiltonian (2.31) contains the parity operator $P = \exp(\mathrm{i}\pi a^\dagger a)$. In the special case $g_0 = g_1$, the Hamiltonian (2.31) reduces to the Hamiltonian

$$H = \omega a^\dagger a + g a^{\dagger 2} + g^* a^2 \tag{2.32}$$

that describes the degenerate down-conversion process in which the usual single-mode squeezed states are produced.

3. Quantum statistical properties

3.1. Photon statistics

The number-state decomposition of the ordinary squeezed states $|\beta; \xi, \lambda\rangle = S(\xi, \lambda)|\beta\rangle$ is given by [1]

$$|\beta; \xi, \lambda\rangle = \frac{1}{\sqrt{\mu}} \exp\left(-\frac{|\beta|^2}{2} + \frac{\nu^*}{2\mu}\beta^2\right) \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{\nu}{2\mu}\right)^{n/2} H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) |n\rangle \tag{3.1}$$

where the parameters μ and ν are defined according to (2.23) and $H_n(z)$ are the Hermite polynomials. By using equation (3.1), we easily obtain the number-state decomposition of the parity-dependent squeezed states:

$$|\beta; \xi_0, \lambda_0; \xi_1, \lambda_1\rangle = \sum_{j=0}^{1} \frac{1}{\sqrt{\mu_j}} \exp\left(-\frac{|\beta|^2}{2} + \frac{\nu_j^*}{2\mu_j}\beta^2\right) \sum_{n=0}^{\infty} \frac{1}{\sqrt{(2n+j)!}} \left(\frac{\nu_j}{2\mu_j}\right)^{(2n+j)/2}$$
$$\times H_{2n+j}\left(\frac{\beta}{\sqrt{2\mu_j\nu_j}}\right) |2n+j\rangle\,. \tag{3.2}$$

Then we find that the photon-number distribution $P(n) = |\langle n|\beta; \xi_0, \lambda_0; \xi_1, \lambda_1\rangle|^2$ is given by

$$P(n) = \exp(-|\beta|^2 + |\beta|^2 \tanh r_j \cos 2\psi_j) \frac{\tanh^n r_j}{2^n n! \cosh r_j} \left| H_n\left(\frac{|\beta|\, \mathrm{e}^{\mathrm{i}\psi_j}}{\sqrt{\sinh 2r_j}}\right)\right|^2 \tag{3.3}$$

where the index j is 0 for even n and 1 for odd n. We use the notation

$$\psi_j \equiv \phi_\beta + \lambda_j + \tfrac{1}{2}\theta_j \tag{3.4}$$
$$\phi_\beta \equiv \arg\beta\,. \tag{3.5}$$

Numerical results are shown in figures 1 and 2. The known oscillations [5] in the photon-number distribution of the ordinary squeezed states also appear in the parity-dependent case. Due to the fact that the distributions for even and odd photon numbers depend on different parameters, these oscillations can be enhanced or decreased by a suitable choice of the parameters. The distribution $P(n)$ of (3.3) shows oscillations of two types: 'slow'

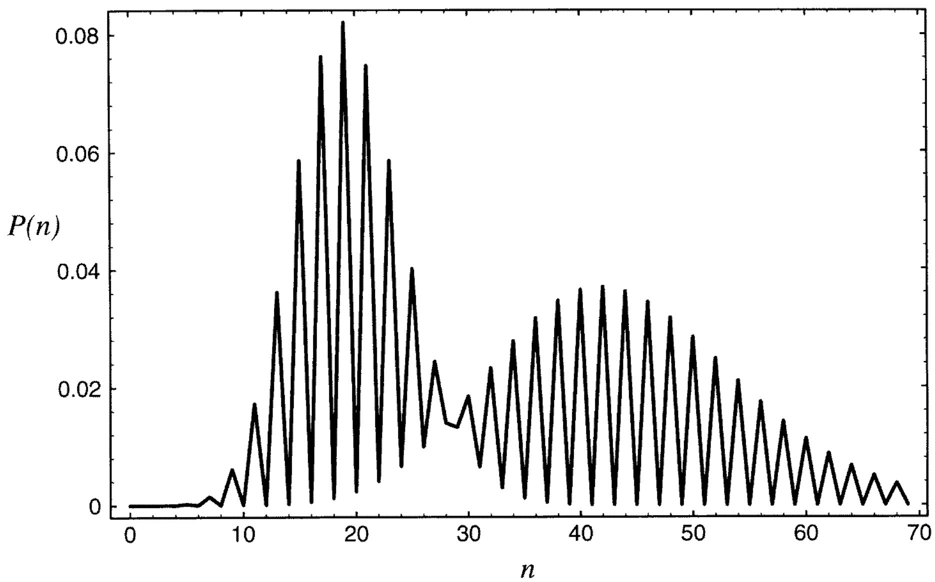

Figure 1. The photon-number distribution $P(n)$ for a parity-dependent squeezed state with $|\beta| = 4$, $r_0 = 0.5$, $\psi_0 = \pi/2$, $r_1 = 0.1$, $\psi_1 = \pi/2$. The wide peaks on the left- and right-hand sides contain sharp peaks for odd and even values of n, respectively.

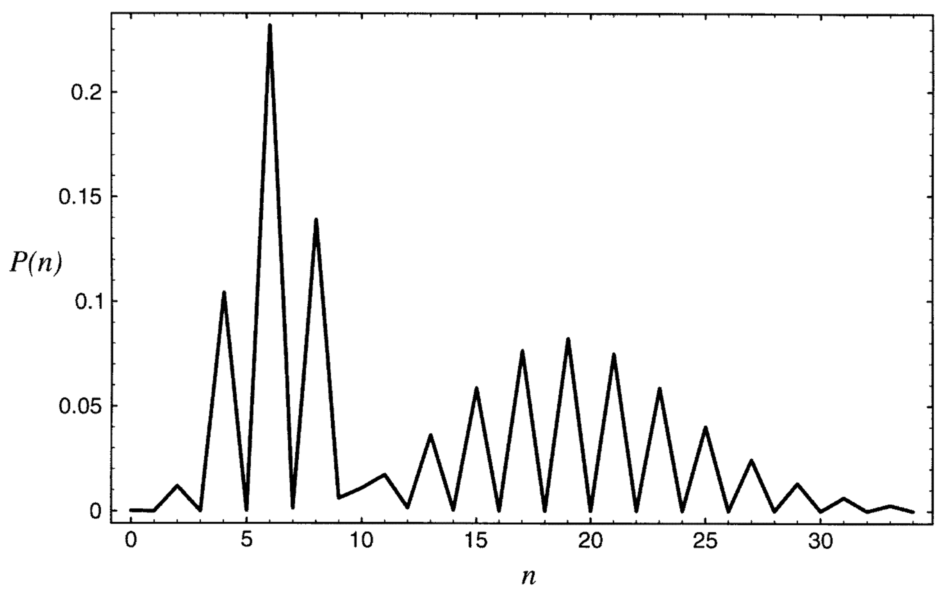

Figure 2. The photon-number distribution $P(n)$ for a parity-dependent squeezed state with $|\beta| = 4$, $r_0 = 0.5$, $\psi_0 = 0$, $r_1 = 0.1$, $\psi_1 = \pi/2$. The wide peaks on the left- and right-hand sides contain sharp peaks for even and odd values of n, respectively.

oscillations that follow from smooth oscillations of the Hermite polynomials, and 'rapid' oscillations (or, more precisely, sharp jumps) between even and odd values of n. These jumps follow from the fact that the Hermite polynomials for even and odd values of n

behave in a different manner. These features also exist for the ordinary squeezed states, but in the parity-dependent case it is possible to change independently the behaviour of the even and odd parts.

The characteristic function

$$F(z) = \sum_{n=0}^{\infty} z^n P(n) \tag{3.6}$$

allows us to calculate all the normally ordered moments of the number operator $N = a^\dagger a$:

$$\langle a^{\dagger p} a^p \rangle = \langle N(N-1)\cdots(N-p+1)\rangle = \left.\frac{\partial^p F}{\partial z^p}\right|_{z=1} . \tag{3.7}$$

Combining equations (3.3) and (3.6) and using summation theorems for the Hermite polynomials [11], we get

$$F(z) = \frac{\mathrm{e}^{-|\beta|^2}}{2} \sum_{j=0}^{1} \tau_j \big[\mathrm{e}^{z\tau_j^2|\beta|^2} + (-1)^j\, \mathrm{e}^{-z\tau_j^2|\beta|^2}\big] \exp\big[|\beta|^2(1-\tau_j^2 z^2)\tanh r_j \cos 2\psi_j\big] \tag{3.8}$$

where we have defined

$$\tau_j \equiv \left(\cosh^2 r_j - z^2 \sinh^2 r_j\right)^{-1/2} . \tag{3.9}$$

Using equations (3.7) and (3.8), we find analytic expressions for the first and the second moments:

$$\langle a^\dagger a\rangle = \tfrac{1}{2}\sum_{j=0}^{1}\big[A_j^{(+)} + (-1)^j\, \mathrm{e}^{-2|\beta|^2} A_j^{(-)}\big] \tag{3.10}$$

$$\langle a^{\dagger 2} a^2\rangle = \tfrac{1}{2}\sum_{j=0}^{1}\big\{\big[\big(A_j^{(+)}\big)^2 + B_j^{(+)}\big] + (-1)^j\, \mathrm{e}^{-2|\beta|^2}\big[\big(A_j^{(-)}\big)^2 + B_j^{(-)}\big]\big\} \tag{3.11}$$

where we have defined

$$A_j^{(\pm)} \equiv \sinh^2 r_j \pm |\beta|^2 \cosh 2r_j - |\beta|^2 \sinh 2r_j \cos 2\psi_j \tag{3.12}$$

$$B_j^{(\pm)} \equiv \sinh^2 r_j \cosh 2r_j \pm 2|\beta|^2 \sinh^2 r_j (1 + 2\cosh 2r_j) - |\beta|^2 \sinh 2r_j (1 + 4\sinh^2 r_j)\cos 2\psi_j . \tag{3.13}$$

The second-order correlation function

$$g^{(2)} = \frac{\langle a^{\dagger 2} a^2\rangle}{\langle a^\dagger a\rangle^2} = \frac{\langle N^2\rangle - \langle N\rangle}{\langle N\rangle^2} \tag{3.14}$$

can be calculated from equations (3.10) and (3.11).

Numerical calculations show that in the case $r_0 = 0$ (only the odd component is squeezed) antibunching is relatively weak. In this case the minimum value of $g^{(2)}$ is approximately 0.75 for $r_1 = 0.3$, $\psi_1 = 0$, $|\beta| \simeq 1$. Much stronger antibunching is obtained for $r_1 = 0$ (only the even component is squeezed). This situation is shown in figure 3. We see that the parity-dependent squeezed states are antibunched for small values of r_0 and $|\beta|$. Also, we find that maximum antibunching is achieved for $\psi_0 = 0$, i.e. for squeezing in the direction of the displacement of the initial coherent state. The dependence of $g^{(2)}$ on ψ_0 is seen from figure 4. When $\psi_0 = 0$, $r_1 = 0$, very strong antibunching can be achieved for very small values of r_0 and $|\beta|$. This is shown in figure 5 where $g^{(2)}$ is presented for the parity-dependent and ordinary squeezed states. We see that the parity-dependent squeezed states with $r_1 = 0$, $\psi_0 = 0$ exhibit stronger antibunching than the ordinary squeezed states with $r = r_0$, $\psi = 0$.

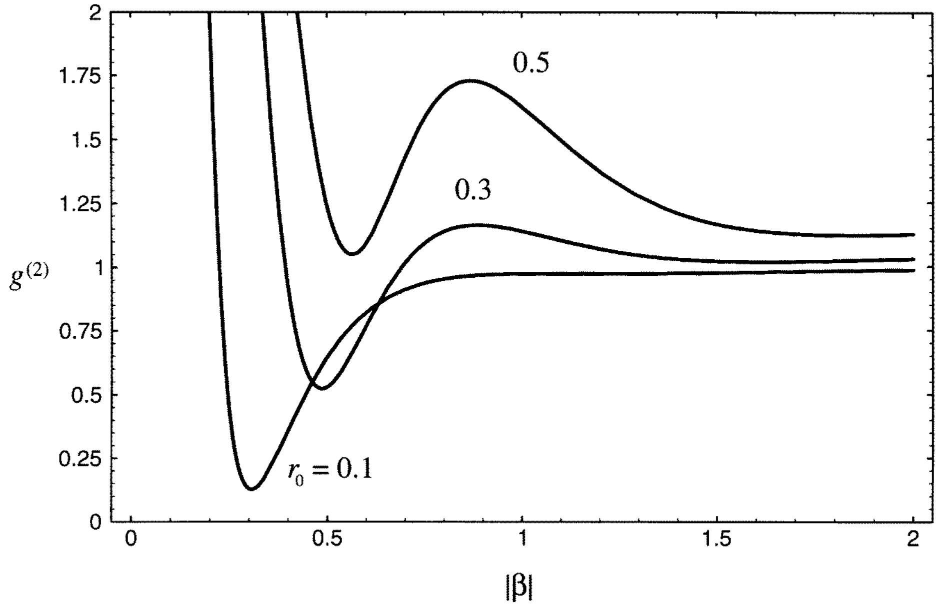

Figure 3. The second-order correlation function $g^{(2)}$ versus $|\beta|$ for $r_1 = 0$, $\psi_0 = \psi_1 = 0$ and various values of r_0. Antibunching appears for $r_0 < 0.48$ and the smaller r_0, the stronger the antibunching.

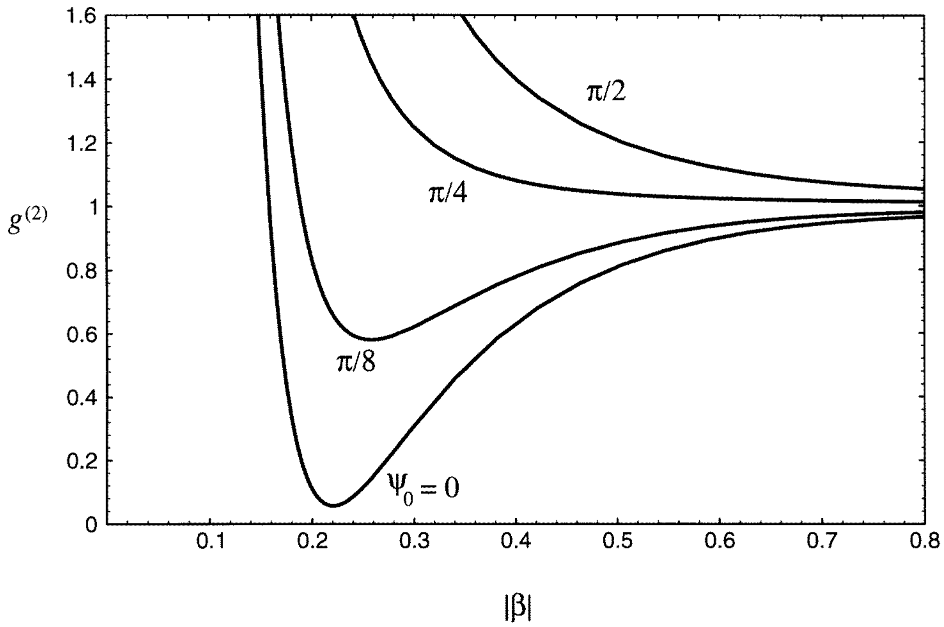

Figure 4. The second-order correlation function $g^{(2)}$ versus $|\beta|$ for $r_0 = 0.05$, $r_1 = 0$, $\psi_1 = 0$ and various values of ψ_0. Antibunching is strong for ψ_0 near zero, but rapidly weakens as ψ_0 increases.

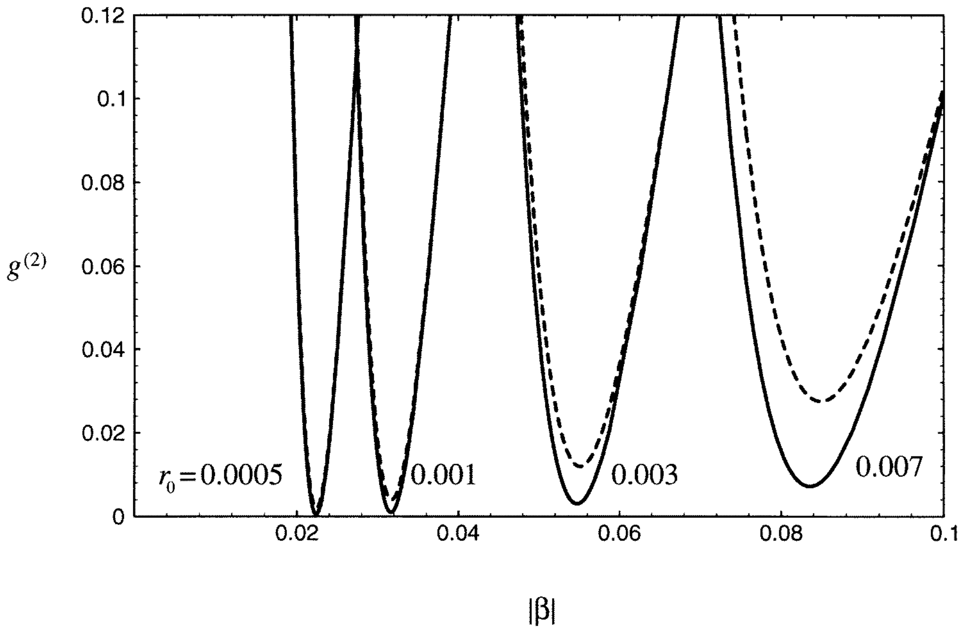

Figure 5. The second-order correlation function $g^{(2)}$ versus $|\beta|$ for parity-dependent squeezed states with $r_1 = 0$, $\psi_0 = 0$ (full curve) and ordinary squeezed states with $r = r_0$, $\psi = 0$ (broken curve). Various values of $r = r_0$ are considered.

3.2. Position and momentum uncertainties

The position and momentum operators are defined as

$$x = \frac{1}{\sqrt{2}}(a^\dagger + a) \tag{3.15}$$

$$p = \frac{\mathrm{i}}{\sqrt{2}}(a^\dagger - a)\,. \tag{3.16}$$

Their variances

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 \tag{3.17}$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 \tag{3.18}$$

obey the Heisenberg uncertainty relation

$$(\Delta x)^2(\Delta p)^2 \geqslant \tfrac{1}{4}\,. \tag{3.19}$$

The x-representation of the parity-dependent squeezed states can be calculated to be

$$\Psi(x) = \langle x|\beta;\xi_0,\lambda_0;\xi_1,\lambda_1\rangle = \frac{1}{2}\left(\frac{1}{\pi}\right)^{1/4}\exp(-|\beta|^2/2)\sum_{j=0}^{1}\frac{1}{\sqrt{\mu_j-\nu_j}}\exp\left(-\frac{\mu_j^*-\nu_j^*}{\mu_j-\nu_j}\frac{\beta^2}{2}\right)$$
$$\times\left[\exp\left(\frac{\sqrt{2}\beta x}{\mu_j-\nu_j}\right)+(-1)^j\exp\left(-\frac{\sqrt{2}\beta x}{\mu_j-\nu_j}\right)\right]\exp\left(-\frac{\mu_j+\nu_j}{\mu_j-\nu_j}\frac{x^2}{2}\right). \tag{3.20}$$

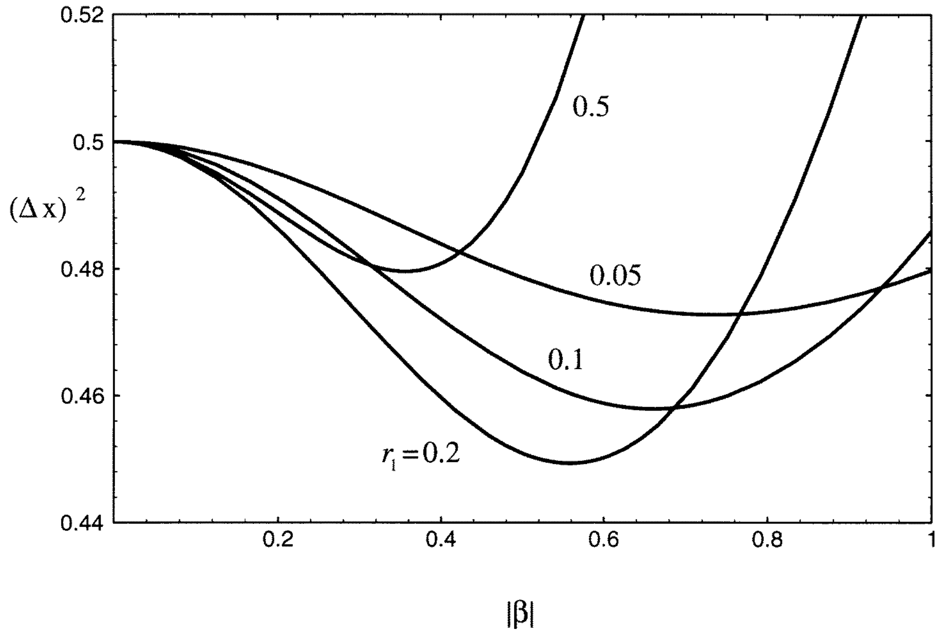

Figure 6. The variance $(\Delta x)^2$ as a function of $|\beta|$ for $\phi_\beta = 0$, $r_0 = 0$, $\theta_0 = \lambda_0 = \theta_1 = \lambda_1 = 0$ and various values of r_1. For moderate values of $|\beta|$ and r_1 the state is squeezed in the x-direction, $(\Delta x)^2 < \frac{1}{2}$.

Moments of the position operator are found by evaluating Gaussian integrals:

$$\langle x \rangle = \frac{\mathrm{e}^{-|\beta|^2}}{2\sqrt{2}} \sum_{j,l=0}^{1} (1 - \delta_{jl})\, \Omega_{jl}^{3/2} \left[V_{jl}^{(+)} \mathrm{e}^{\Omega_{jl}|\beta|^2} + (-1)^j V_{jl}^{(-)} \mathrm{e}^{-\Omega_{jl}|\beta|^2} \right] \times \exp\left\{ \mathrm{i}\Omega_{jl} \mathrm{Im}\left[\beta^{*2} (\mu_j \nu_l - \mu_l \nu_j) \right] \right\} \tag{3.21}$$

$$\langle x^2 \rangle = \frac{1}{4} \sum_{j=0}^{1} \left\{ \left(|\mu_j - \nu_j|^2 + \left[V_{jj}^{(+)} \right]^2 \right) + (-1)^j \, \mathrm{e}^{-2|\beta|^2} \left(|\mu_j - \nu_j|^2 + \left[V_{jj}^{(-)} \right]^2 \right) \right\} \tag{3.22}$$

where δ_{jl} is the Kronecker symbol, and we have defined

$$\Omega_{jl} \equiv (\mu_j \mu_l^* - \nu_j \nu_l^*)^{-1} \tag{3.23}$$

$$V_{jl}^{(\pm)} \equiv \beta^* (\mu_j - \nu_j) \pm \beta (\mu_l^* - \nu_l^*)\,. \tag{3.24}$$

The p-representation of the parity-dependent squeezed states and moments of the momentum operator can be obtained analogously. Then we can also calculate the uncertainty product $(\Delta x)^2 (\Delta p)^2$.

We have studied numerically the behaviour of the variance $(\Delta x)^2$ for $\phi_\beta = \lambda_0 = \lambda_1 = 0$. Calculations show that the parity-dependent squeezed states are squeezed in the x-direction for $\theta_0 = \theta_1 = 0$. Squeezing in the p-direction is obtained, accordingly, for $\theta_0 = \theta_1 = \pm\pi$. In the case $r_1 = 0$, $\theta_0 = 0$ we find that $(\Delta x)^2 \to \frac{1}{2} \exp(-2r_0)$ as $|\beta| \to 0$. As $|\beta|$ increases, $(\Delta x)^2$ increases too. In the case $r_0 = 0$, $\theta_1 = 0$ the situation is essentially different, as shown in figure 6. For small values of $|\beta|$ ($|\beta| \ll 1$), $(\Delta x)^2$ approaches the coherent-state value $\frac{1}{2}$. As $|\beta|$ increases, $(\Delta x)^2$ at first decreases below $\frac{1}{2}$ (squeezing), reaches a minimum and then increases monotonically. The uncertainty product $(\Delta x)^2 (\Delta p)^2$ is plotted in figure 7

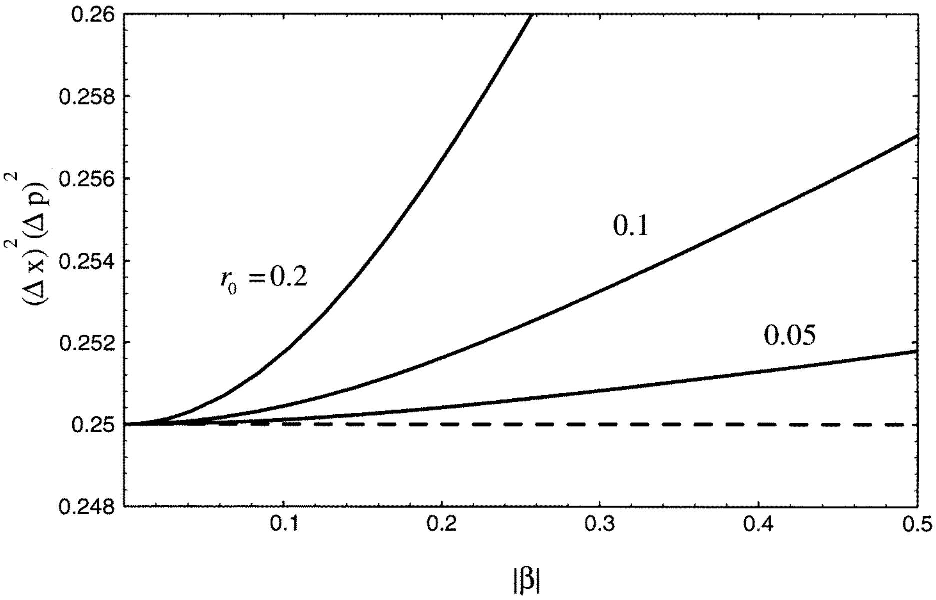

Figure 7. The uncertainty product $(\Delta x)^2(\Delta p)^2$ as a function of $|\beta|$ for $\phi_\beta = 0$, $r_1 = 0$, $\theta_0 = \lambda_0 = \theta_1 = \lambda_1 = 0$ and various values of r_0. The broken line is the minimum available value $\frac{1}{4}$.

as a function of $|\beta|$ for the case $\phi_\beta = 0$, $r_1 = 0$, $\theta_0 = \lambda_0 = \theta_1 = \lambda_1 = 0$ and various values of r_0. The uncertainty product is always greater than its minimum allowed value $\frac{1}{4}$. This value is achieved only in the limit $|\beta| \to 0$. Recall that the uncertainty product in the variables x_b, p_b is always $\frac{1}{4}$.

4. Q and Wigner functions

Useful information about the field state can be inferred from phase-space quasiprobability distributions. We start from the $Q(\alpha)$ function:

$$Q(\alpha) = \frac{1}{\pi} |\langle \alpha|\beta; \xi_0, \lambda_0; \xi_1, \lambda_1\rangle|^2 . \tag{4.1}$$

A straightforward calculation gives

$$\langle \alpha|\beta; \xi_0, \lambda_0; \xi_1, \lambda_1\rangle = \tfrac{1}{2}\, \mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2} \sum_{j=0}^{1} \frac{1}{\sqrt{\mu_j}} \left[\mathrm{e}^{\alpha^*\beta/\mu_j} + (-1)^j\, \mathrm{e}^{-\alpha^*\beta/\mu_j}\right] \times \exp\left(\frac{\nu_j^*}{2\mu_j}\beta^2 - \frac{\nu_j}{2\mu_j}\alpha^{*2}\right) . \tag{4.2}$$

The Wigner function is given by [12]

$$W(x, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Psi(x+s)\Psi^*(x-s)\, \mathrm{e}^{-2\mathrm{i}ps}\, \mathrm{d}s . \tag{4.3}$$

Using equation (3.20) for the x-representation wavefunction and evaluating the Gaussian integrals, we find the following result:

$$
\begin{aligned}
W(x,p) = \frac{1}{4\pi} \sum_{j,l=0}^{1} \Omega_{jl}^{1/2} \exp\left(-|\beta|^2 - Z_{jl} - T_{jl}\frac{x^2}{2}\right) \\
\times \left\{ (-1)^j \exp\left[\frac{\left(R_{jl}x - 2\mathrm{i}p - K_{jl}\right)^2}{2T_{jl}} + L_{jl}x\right] \right. \\
+(-1)^l \exp\left[\frac{\left(R_{jl}x - 2\mathrm{i}p + K_{jl}\right)^2}{2T_{jl}} - L_{jl}x\right] \\
+(-1)^{j+l} \exp\left[\frac{\left(R_{jl}x - 2\mathrm{i}p + L_{jl}\right)^2}{2T_{jl}} - K_{jl}x\right] \\
\left. + \exp\left[\frac{\left(R_{jl}x - 2\mathrm{i}p - L_{jl}\right)^2}{2T_{jl}} + K_{jl}x\right]\right\}
\end{aligned}
\tag{4.4}
$$

where we have defined

$$Z_{jl} \equiv \frac{\mu_j^* - \nu_j^*}{\mu_j - \nu_j}\frac{\beta^2}{2} + \frac{\mu_l - \nu_l}{\mu_l^* - \nu_l^*}\frac{\beta^{*2}}{2} \tag{4.5}$$

$$T_{jl} \equiv \frac{\mu_j + \nu_j}{\mu_j - \nu_j} + \frac{\mu_l^* + \nu_l^*}{\mu_l^* - \nu_l^*} \tag{4.6}$$

$$R_{jl} \equiv -\frac{\mu_j + \nu_j}{\mu_j - \nu_j} + \frac{\mu_l^* + \nu_l^*}{\mu_l^* - \nu_l^*} \tag{4.7}$$

$$K_{jl} \equiv \frac{\sqrt{2}\beta}{\mu_j - \nu_j} + \frac{\sqrt{2}\beta^*}{\mu_l^* - \nu_l^*} \tag{4.8}$$

$$L_{jl} \equiv -\frac{\sqrt{2}\beta}{\mu_j - \nu_j} + \frac{\sqrt{2}\beta^*}{\mu_l^* - \nu_l^*} \tag{4.9}$$

and Ω_{jl} is defined by equation (3.23). We see that in the parity-dependent case both the Q and Wigner functions are given by a superposition of a number of Gaussians and not by a single Gaussian as in the ordinary case.

For the parity-dependent squeezing the interference in phase space produces Q and Wigner functions of interesting forms. Figure 8 shows that the $Q(\alpha)$ function for the case of a strongly squeezed even component is similar to that of a number eigenstate. The $Q(\alpha)$ functions shown in figures 9 and 10 have $r_0 = r_1$ and $\lambda_0 = \lambda_1$, but $\theta_0 \neq \theta_1$. And yet this is enough to split the Gaussian Q function of an ordinary squeezed state into three Gaussians in figure 9 and five Gaussians in figure 10. Some examples of the Wigner function for the parity-dependent squeezed states are shown in figures 11 and 12. In figure 11 we see a big peak along the line $p = 0$ and a smaller 'wave' along the line $x = 0$. When these two structures intersect near the origin, two sharp negative peaks are produced. Besides, two high positive peaks become at the intersection of the big peak with two smaller Gaussians perpendicular to it. In figure 12 a very impressive interference occurs along the line $x = 0$ where sharp positive and negative peaks alternate.

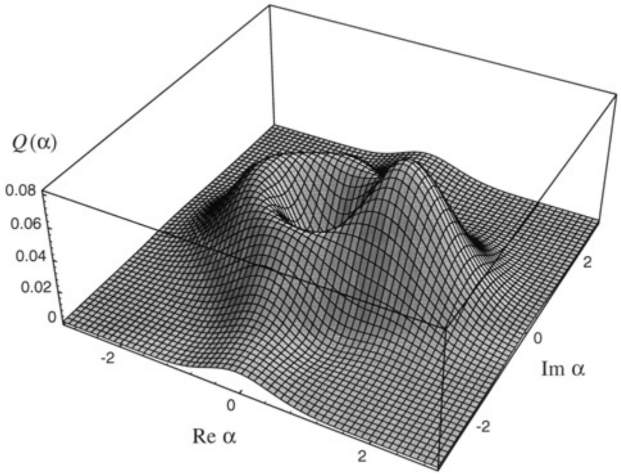

Figure 8. The function $Q(\alpha)$ for a parity-dependent squeezed state with $|\beta| = 1$, $\phi_\beta = 0$, $r_0 = 4$, $r_1 = 0$, $\theta_0 = \theta_1 = 0$, $\lambda_0 = \lambda_1 = 0$.

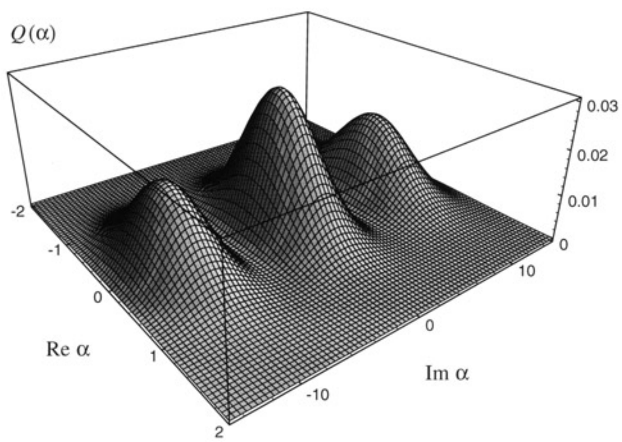

Figure 9. The function $Q(\alpha)$ for a parity-dependent squeezed state with $|\beta| = 3$, $\phi_\beta = 0$, $r_0 = r_1 = 3$, $\theta_0 = 0$, $\theta_1 = \pi$, $\lambda_0 = \lambda_1 = 0$.

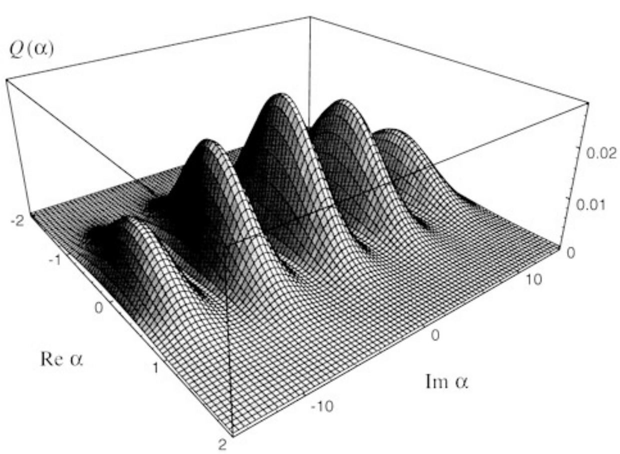

Figure 10. The function $Q(\alpha)$ for a parity-dependent squeezed state with $|\beta| = 5$, $\phi_\beta = 0$, $r_0 = r_1 = 3$, $\theta_0 = 0$, $\theta_1 = \pi$, $\lambda_0 = \lambda_1 = 0$.

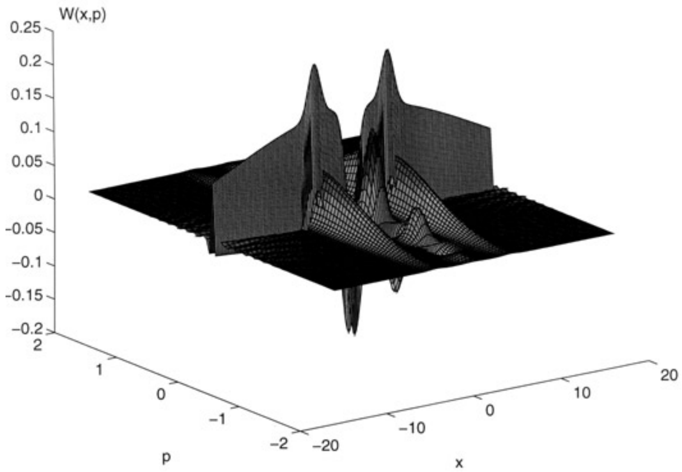

Figure 11. The Wigner function for a parity-dependent squeezed state with $|\beta| = 3$, $\phi_\beta = 0$, $r_0 = 3$, $r_2 = 0$, $\theta_0 = \pi$, $\theta_1 = 0$, $\lambda_0 = \lambda_1 = 0$.

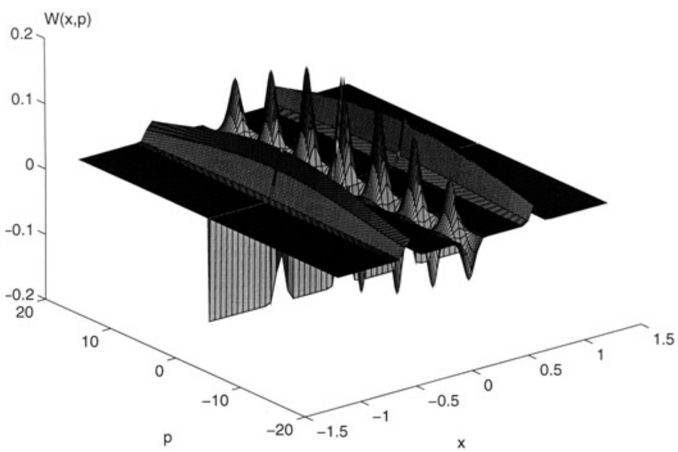

Figure 12. The Wigner function for a parity-dependent squeezed state with $|\beta| = 8$, $\phi_\beta = 0$, $r_0 = r_1 = 3$, $\theta_0 = 0$, $\theta_1 = \pi$, $\lambda_0 = \lambda_1 = 0$.

5. Conclusions

In this paper we have introduced the concept of parity-dependent squeezing for the single-mode light field. It is based on the fact that squeezing transformations are elements of the $SU(1,1)$ Lie group, and therefore we proposed a squeezing operator that acts differently on distinct irreducible representations of $SU(1,1)$. For the case of single-mode squeezing this operator is parity dependent. We have considered the parity-dependent Bogoliubov transformations and parity-dependent Bogoliubov quasiparticles. A parity-dependent quadratic Hamiltonian has been given that evolves coherent states into the parity-dependent squeezed states. Quantum statistical properties of these states have been studied in detail. We have found interesting non-classical features such as strong oscillations in the photon-number distribution, strong antibunching and quadrature squeezing. Results for the Q and Wigner functions show that parity-dependent squeezing considered in this paper leads to very interesting interference effects in phase space which are absent in ordinary squeezing.

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